

Framework for Mathematical Proficiency for Teaching

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Secondary school mathematics comprises far more than facts, routines, and strategies. It includes a vast array of interrelated mathematical concepts, ways to represent and communicate those concepts, and tools for solving all kinds of mathematical problems. It requires reasoning and creativity, providing learners with mathematical knowledge while also laying a foundation for further studies in mathematics and other disciplines.

To facilitate the learning of secondary school mathematics, teachers need a particular kind of proficiency. Mathematical proficiency for teaching at the secondary level requires that teachers know not only more mathematics than they teach but also know it more deeply. Furthermore, mathematical proficiency for teaching is different from that needed in other professions. Mathematical proficiency for teaching is different from the mathematical proficiency needed for engineering, accounting, or the medical professions. It is even different from the mathematical proficiency a mathematician needs.

Mathematical proficiency for teaching (MPT) is not the same as proficiency in pedagogy. Being equipped to be an effective mathematics teacher is not simply a matter of “knowing the mathematics” plus “knowing how to teach.” The task of teaching mathematics cannot be partitioned into such simple categories. Teaching mathematics requires a special kind of mathematical proficiency that is unique to the teaching profession.

Secondary school mathematics clearly differs from elementary school mathematics in many ways, and MPT at the secondary level is similarly different from the MPT required to successfully teach elementary school mathematics. For example, the mathematical structures of secondary mathematics are different from those addressed in elementary school, and secondary teachers must have a firm grasp of issues such as the importance of defining the domain under consideration. An elementary school teacher, for example, need not explicitly restrict the domain of an operation to positive integers only, but a secondary teacher might need to do so. A secondary teacher must also understand justification and proof differently from how an elementary teacher does. Students do not encounter axiomatic systems or the logic and rigor required to construct a valid proof until after elementary school, but what qualifies as proof in a third-grade classroom is different from what qualifies as proof in a tenth-grade geometry course. Another example of a qualitative difference between MPT at the secondary level and MPT at the elementary level concerns levels of abstraction. The content of secondary mathematics requires that teachers be able both to think abstractly and to abstract ideas from concrete mathematical objects and operations.

A Framework for MPT

Mathematical proficiency for teaching (MPT) can be viewed from three perspectives or through three lenses: mathematical proficiency, mathematical activity, and mathematical work of teaching. Each perspective provides a window through which MPT can be seen (Figure 1). The multi-dimensional nature of MPT is part of what sets it apart from the mathematical proficiency required in fields like engineering, architecture, and research mathematics. Furthermore, the fact that these windows are not bounded indicates that (1) we do not claim to provide an exhaustive description of each facet of MPT, and (2) MPT grows and develops throughout a teacher's career. MPT is a quality and not an endpoint.

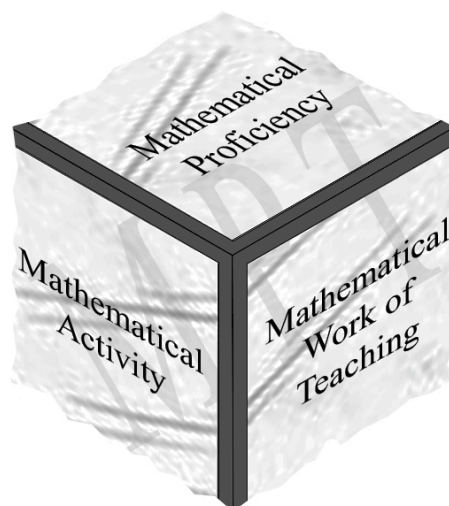


Figure 1. Mathematical proficiency for teaching.

Mathematical proficiency includes aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students. The mathematical proficiency teachers need, however, goes well beyond what one might find in secondary students. The students' development of mathematical proficiency usually depends heavily on how well developed the teacher's proficiency is.

Mathematical activity can be thought of as “doing mathematics.” Examples include representing mathematical objects and operations, connecting mathematical concepts, modeling mathematical phenomena, and justifying mathematical arguments. This facet of teachers' mathematical proficiency is on display as they engage students in the day-to-day study of mathematics. Teachers need deep knowledge, for example, of what characterizes the structure of mathematics (as opposed to conventions that have been adopted over the centuries) and how to generalize mathematical findings. The more a teacher's proficiency in mathematical activity has developed, the better equipped he or she will be to facilitate the learning and doing of mathematics.

The *mathematical work of teaching* diverges sharply from the mathematical proficiency needed in other professions requiring mathematics. One of its aspects is an

understanding of the mathematical thinking of students, which may include, for example, recognizing the mathematical nature of their errors and misconceptions. Another aspect of the mathematical work of teaching is knowledge of and proficiency in the mathematics that comes before and after what is being studied currently. A teacher benefits from knowing what students have learned in previous years so that he or she can help them build upon that prior knowledge. The teacher also needs to provide a foundation for the mathematics they will be learning later, which requires knowing and understanding the mathematics in the rest of the curriculum.

The three components of MPT--mathematical proficiency, mathematical activity, and mathematical work of teaching--together form a full picture of the mathematics required of a teacher of secondary mathematics. It is not enough to know the mathematics that students are learning. Teachers must also possess a depth and extent of mathematical proficiency that will equip them to foster their students' mathematical proficiency. Mathematical proficiency informs the other two perspectives on MPT: Mathematical activity and the mathematical work of teaching emerge from, and depend upon, the teacher's mathematical proficiency.

An Example of MPT Use

In responding to the following situation, no matter how it is handled pedagogically, the teacher needs to make use of all facets of his or her MPT:

In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \cdot x^n = x^5$ on the board and asked the students to make a list of values for m and n that made the statement true. After a few minutes, one student asked, "Can we write them all down? I keep thinking of more."

To decide whether the student's question is worth pursuing, frame additional questions appropriately, and know how to proceed from there, the teacher needs conceptual understanding and productive disposition (two aspects of mathematical proficiency). The concept of an exponent is more complicated than might be initially apparent. Does the rule $x^m \cdot x^n = x^{m+n}$ always apply? Must the domain of x be restricted? Must the domain of m and n be restricted? These are questions the teacher needs sufficient mathematical proficiency to address. With respect to mathematical activity, the teacher's proficiency in representing exponents, knowing constraints that may be helpful in dealing with them, and making connections between exponents and other mathematical phenomena are all crucial to successfully teaching the concept. What are the advantages of a graphical representation of an exponential function as opposed to a symbolic representation? How is the operation of exponentiation connected to the operation of multiplication? Does an exponent always indicate repeated multiplication? With respect to the mathematical work of teaching, it is critical that the teacher knows and understands the mathematics that typically comes before and after the point in the curriculum where a problem like the one involving the rule $x^m \cdot x^n$ is addressed. For example if this problem is being discussed in a beginning algebra course, it is important to realize that students have probably had limited exposure to

exponents and may think about them only in terms of the repeated multiplication of natural numbers. And to lay a good foundation for later studies of exponential functions, the teacher needs to know that there may be discontinuity in the graph of x^n depending on the domain of both the base and the exponent.

Elaboration of the MPT Perspectives

An outline of our framework for the three perspectives on MPT is shown in Figure 2. In this section, we amplify each perspective in turn.

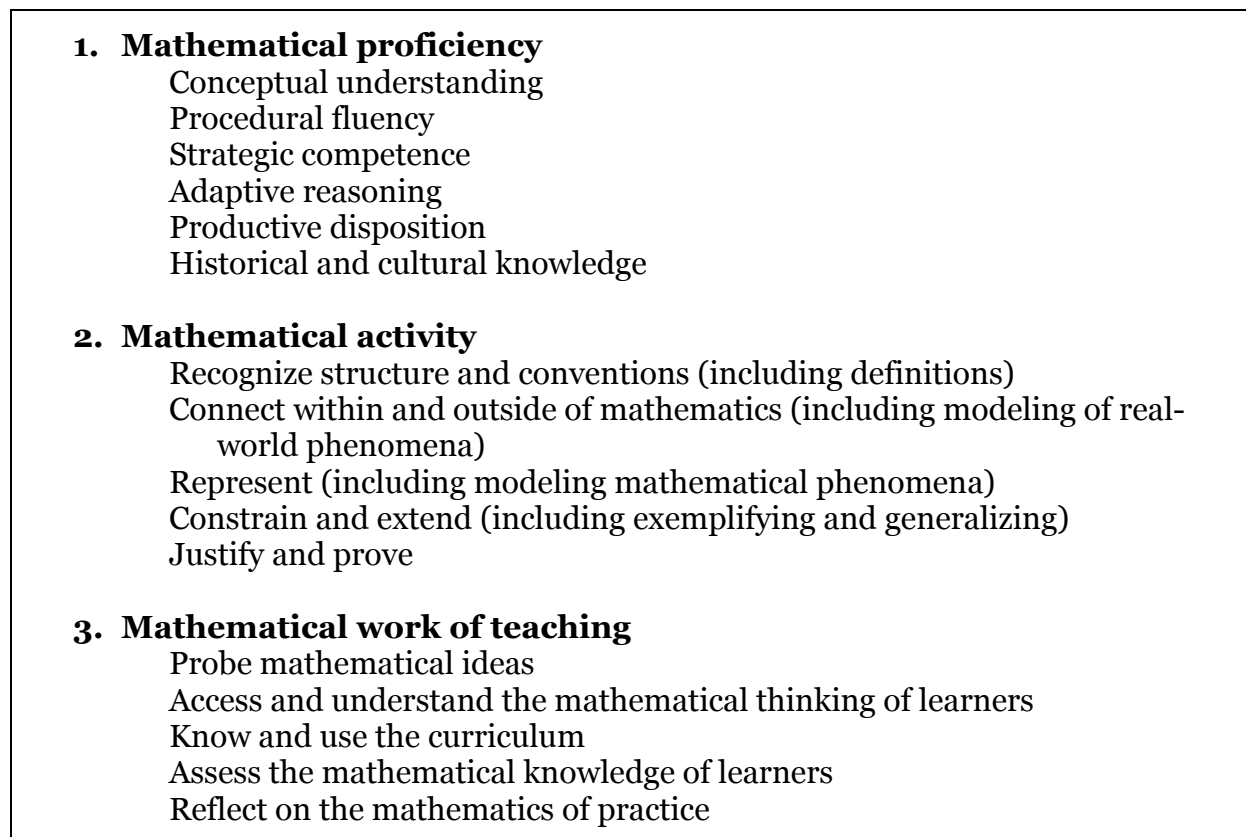


Figure 2. Framework for mathematical proficiency for teaching (MPT).

The philosopher Gilbert Ryle (1949) claimed that there are two types of knowledge: The first is expressed as “knowing that,” sometimes called *propositional* or *factual* knowledge, and the second as “knowing how,” sometimes called *practical* knowledge. Because we wanted to capture this distinction and at the same time to enlarge the construct of *mathematical knowledge for teaching* to include such mathematical aspects as reasoning, problem solving, and disposition, we have adopted the term *proficiency* throughout this document instead of using *knowledge*. We use *proficiency* in much the same way as it is used in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) except that we are applying it to teachers rather than students and to their teaching as well as to their knowing and doing of mathematics.

Mathematical Proficiency

The principal goal of secondary school mathematics is to develop all facets of the learners' mathematical proficiency, and the teacher of secondary mathematics needs to be able to help students with that development. Such proficiency on the teacher's part requires that the teacher not only understand the substance of secondary school mathematics deeply and thoroughly but also know how to guide students toward greater proficiency in mathematics. We have divided the teacher's mathematical proficiency into six strands, shown in Figure 2, to capture the multifaceted nature of that proficiency.

There is a range of proficiency in each strand, and a teacher may become increasingly proficient in the course of his or her career. At the same time, certain categories may involve greater depth of mathematical knowledge than others. For example, *conceptual understanding* involves a different kind of knowledge than *procedural fluency*, though both are important. Only rote knowledge is required in order to demonstrate procedural fluency in mathematics. Conceptual understanding, however, involves (among other things) knowing *why* the procedures work.

Conceptual understanding. Conceptual understanding is sometimes described as the “knowing *why*” of mathematical proficiency. A person may demonstrate conceptual understanding by such actions as deriving needed formulas without simply retrieving them from memory, evaluating an answer for reasonableness and correctness, understanding connections in mathematics, or formulating a proof.

Some examples of conceptual understanding are the following:

1. Knowing and understanding where the quadratic formula comes from (including being able to derive it),
2. Seeing the connections between right triangle trigonometry and the graphs of trig functions, and
3. Understanding how the introduction of an outlying data point can affect mean and median differently.

Procedural fluency. A person with procedural fluency knows some conditions for when and how a procedure may be applied and can apply it competently. Procedural fluency alone, however, would not allow one to independently derive new uses for a previously learned procedure, such as completing the square to solve $ax^6 + bx^3 = c$. Procedural fluency can be thought of as part of the “knowing *how*” of mathematical proficiency. Such fluency is useful because the ability to quickly recall and accurately execute procedures significantly aids in the solution of mathematical problems.

The following are examples of procedural fluency:

1. Recalling and using the algorithm for long division of polynomials,
2. Sketching the graph of a linear function,
3. Finding the area of a polygon using a formula, and
4. Using key words to translate the relevant information in a word problem into an algebraic expression.

Strategic competence. Strategic competence requires procedural fluency as well as a certain level of conceptual understanding. Demonstrating strategic competence requires the ability to generate, evaluate, and implement problem-solving strategies. That is, a person must first be able to generate possible problem-solving strategies (such as utilizing a known formula, deriving a new formula, solving a simpler problem, trying extreme cases, or graphing), and then must evaluate the relative effectiveness of those strategies. The person must then accurately implement the chosen strategy. Strategic competence could be described as “knowing how,” but it is different from procedural fluency in that it requires creativity and flexibility because problem-solving strategies cannot be reduced to mere procedures.

Specific examples of strategic competence are the following:

1. Recognizing problems in which the quadratic formula is useful (which goes beyond simply recognizing a quadratic equation or function), and
2. Figuring out how to partition a variety of polygons into “helpful” pieces so as to find their areas.

Adaptive reasoning. A teacher or student with adaptive reasoning is able to recognize current assumptions and adjust to changes in assumptions and conventions. Adjusting to these changes involves comparing assumptions and working in a variety of mathematical systems. For example, since they are based on different assumptions, Euclidean and spherical geometries are structurally different. A person with adaptive reasoning, when introduced to spherical geometry, would consider the possibility that the interior angles of a triangle do not sum to 180° . Furthermore, he or she would be able to construct an example of a triangle, within the assumptions of spherical geometry, whose interior angles sum to more than 180° .

Adaptive reasoning includes the ability to reason both formally and informally. Some examples of formal reasoning are using rules of logic (necessary and sufficient conditions, syllogisms, etc.) and structures of proof (by contradiction, induction, etc.). Informal reasoning may include creating and understanding appropriate analogies, utilizing semi-rigorous justification, and reasoning from representations.

Examples of adaptive reasoning are as follows:

1. Recognizing that division by an unknown is problematic,
2. Working with both common definitions for a trapezoid,
3. Operating in more than one coordinate system,
4. Proving an if-then statement by proving its contrapositive, and
5. Determining the validity of a proposed analogy.

Productive disposition. Those people with a productive disposition believe they can benefit from engaging in mathematical activity and are confident that they can succeed in mathematical endeavors. They are curious and enthusiastic about mathematics and are therefore motivated to see a problem through to its conclusion, even if that involves thinking about the problem for an extended time so as to make progress. People with a productive disposition are able to notice mathematics in the

world around them and apply mathematical principles to situations outside the mathematics classroom. They possess Cuoco's (1996) "habits of mind."

Examples of productive disposition are as follows:

1. Noticing symmetry in the natural world,
2. Persevering through multiple attempts to solve a problem, and
3. Taking time to write and solve a system of equations for comparing phone service plans.

Historical and cultural knowledge. Having knowledge about the history of mathematics is beneficial for many reasons. One prominent benefit is that a person with such knowledge will likely have a deeper understanding of the origin and significance of various mathematical conventions, which in turn may increase his or her conceptual understanding of mathematical ideas. For example, knowing that the integral symbol \int is an elongated s , from the Latin *summa* (meaning *sum* or *total*) may provide a person with insight about what the integral function is. Some other benefits of historical knowledge include an awareness of which mathematical ideas have proven the most useful in the past, an increased ability to predict which mathematical ideas will likely be of use to students in the future, and an appreciation for current developments in mathematics.

Cross-cultural knowledge (i.e., awareness of how people in various cultures or even in various disciplines conceptualize and express mathematical ideas) may have a direct impact on a person's mathematical understanding. For example, a teacher or student may be used to defining a rectangle in terms of its sides and angles, but people in some non-Western cultures define a rectangle in terms of its diagonals. Being able to conceptualize both definitions can strengthen one's mathematical proficiency.

The following are additional examples of historical and cultural knowledge:

1. Being familiar with the historic progression from Euclidean geometry to multiple geometric systems,
2. Being able to compare mathematicians' convention of measuring angles counterclockwise from horizontal with the convention (used by pilots, ship captains, etc.) of indicating directions in terms of degrees clockwise from North,
3. Understanding similarities and differences in algorithms typically taught in North America and those taught elsewhere,
4. Knowing that long-standing "open problems" in mathematics continue to be solved and new problems posed, and
5. Recognizing the increasing use of statistics in the business world.

Mathematical Activity

Underpinning knowledge of mathematical ideas are the processes that evidence such knowledge and the objects on which those processes are performed. The mathematical activity perspective comprises mathematical processes and the

mathematical objects that are the targets of those processes.¹ Our work over the past few years has centered on identifying the opportunities and venues available to a secondary teacher to call on his or her mathematical proficiency in the service of teaching and on developing descriptions of the mathematics that might be called on in each of those settings. We call these descriptions *situations*. Through our analysis of the situations we developed, we have identified general types of mathematical activities that underpin the mathematics that secondary mathematics teachers can productively use in their teaching. Through a perspective on mathematical activity, we acknowledge that mathematical knowledge has a dynamic aspect by describing actions taken upon mathematical objects. Mathematical objects include functions, numbers, matrices, and so on. One might think of them as the nouns of mathematics. The categories in the dimension of mathematical activity describe the verbs of secondary mathematics teaching—the actions one uses with these mathematical objects.

Recognize structure and conventions. Prior to instruction on algebra, school mathematics (usually presecondary) deals with integers and rational numbers. These sets have their own algebraic structure, and students in presecondary classrooms are exposed to those structures through their interaction with systems of whole numbers and fractions. At the secondary level, the rate of introduction of new mathematical systems increases, and the need to account for differences in structure of different mathematical systems becomes more pronounced. The structure of algebra is revealed as students move from the study of rational numbers to the study of real and complex numbers, variables, and functions. Operations once performed only on rational numbers are extended to new objects such as polynomials. New entities such as the inverse function and composition of functions are introduced. In geometry, analytic and other non-Euclidean geometries are introduced. With each new set of operations and numbers come new properties. Secondary mathematics teachers need to be comfortable with differences in properties among mathematical systems so that they can help their students focus on the structure rather than solely on the procedures used in working within that structure. With new structures come new properties and new conventions. In algebra, new notations such as $(f \circ g)(x)$ and summation notation succinctly portray both processes and objects. Familiar notation such as the exponent -1 (e.g., x^{-1} and f^{-1}) is used in different ways depending upon the context. Teachers who recognize similar notation and who are aware of different meanings for notation that appears to be the same need also to be able to identify and explain the conditions under which particular meanings for the notation are appropriate. Definitions are conventions as well. Secondary mathematics teachers with a refined perspective on definition will be able to lead their students in developing sophisticated mathematical arguments involving the mathematical object being defined.

Secondary mathematics teachers draw on their ability to recognize and distinguish between mathematical properties, constraints, or structure in a given mathematical entity or setting, or across instances of a mathematical entity. By

¹ This notion of viewing mathematical proficiency through the lens of mathematical processes and products has underpinned the work of the Mid-Atlantic Center on investigating the mathematical knowledge of teachers and how they draw on that knowledge in their teaching (Zbiek, Conner, & Peters, 2008).

recognizing structural similarity or differences, they seek and make connections between (features of) representations of the same mathematical object or different methods for solving problems (e.g., they recognize the structural similarities in the Euclidean algorithm and the long division algorithm), between mathematical objects of different classes (e.g., they recognize that properties of a set are not inherited when the set is extended to a superset), or between the same objects in different systems (e.g., they recognize the difference in solutions when solving an equation in the real number system and in the complex number system). They recognize conventions in notation and distinguish among the meanings of notations that are similar in appearance. It is in the secondary grades that this attention to structure and convention begins to emerge in school mathematics curricula. Teachers of secondary mathematics can draw on their knowledge of structure and convention to direct students' attention to the nuances of working in different systems and communicating about results in those systems.

Secondary algebra teachers apply their knowledge of algebraic structure (e.g., field properties, properties of equivalence relations, and properties of equality) coupled with their ability to think about their students' mathematical thinking to organize instruction on algebraic transformations.² Recognizing algebraic structures allows teachers to identify potential symbolic rules and to test them. For example, teachers who are aware that the truth of $f(a) + f(b) = f(a + b)$ depends on the nature of the function f , and that students tend to apply this "student's distributive property" indiscriminately, can alert their students to the need to be wary of overgeneralization of this linearity property and test the validity of each potential rule. Teachers who are aware that familiar operations do not have the same meaning when applied to different mathematical objects and structures will know not to generalize properties of multiplication over the set of real numbers to multiplication over the set of matrices, for example. They will also know that they need to exercise caution in extending the rules of exponents, developed and proved for natural number exponents, to negative, rational, real, or complex exponents. Secondary mathematics teachers' recognition of structure underpins their ability to determine how two classes of mathematical objects are related. This ability allows them to recognize the similarity of the structure of switching circuits, conjunction and disjunction of sets, and Boolean algebra, and to apply the properties of Boolean algebras to these other settings.

Teachers of secondary mathematics use definitions in their daily work. They need to be able to appeal to a definition to resolve mathematical questions, and they need to be able to reason from a definition. Less frequently, teachers need to create definitions and to assess the definitions that students create or propose. Creating a definition requires identifying and articulating a combination of a set of characteristics and the relationships among these characteristics in such a way that the combination can be used to determine whether an object, action, or idea belongs to a class of objects, actions, or ideas.

² Algebraic transformations such as the production of equivalent expressions and equivalent equations are the core of many school algebra courses.

An example of the importance of definition occurred in the context of work with a symbolic manipulation calculator. When a seventh grader used a symbolic calculator to evaluate the function $f(x) = \left| \sqrt{x-10} \right|$ at $x = -5$, he was surprised to see that the calculator returned a real value (Heid, Hollebrands, & Iseri, 2002). A teacher with knowledge of a modulus definition of absolute value could have steered the student to reconsider his working definition of absolute value as the “positive value” of a number.

Connect within and outside mathematics. Secondary mathematics introduces more content from more branches of mathematics than mathematics in the elementary grades. Systematic study of geometry, algebra, statistics, probability, discrete mathematics, and the calculus extends students’ mathematical knowledge, and teachers find it useful to be able to make connections among principal mathematical generalizations, definitions, and objects across these areas of mathematics. Connecting within mathematics requires teachers to have a working knowledge of both the characteristics and structure of the mathematics they are teaching and how that mathematical topic relates to other areas of mathematics. For example, students may study transformations using paper folding to investigate reflections, rotations, translations, and glide reflections. When students are studying the Cartesian plane, the teacher should be able to quantify the transformations. Matrices and matrix operations can be used to transform one figure into another. Connecting within mathematics also means being able to connect student-generated algorithms to the standard algorithm. Rewriting equations from the Cartesian coordinate system into polar coordinates goes beyond being able to work in both systems and illustrating the similarities between the two. Teaching even elementary algebra requires of teachers not only the recognition of the Cartesian connection, the knowledge that the set of points that lie on the graph of a function such as $f(x) = 3x^2 - 8x + 2$ are the entire set of (x, y) coordinates that satisfy the equation, but also the knowledge that this connection is not an easy let alone automatic one for students to make. Teachers can provide rich and challenging environments for their students when they are able to move smoothly from question to question, both fielding student questions and posing challenges that require students to connect mathematical ideas. For example, a teacher might ask: “What sort of numbers do different expressions in x generate as the domain of x changes? Why is the product of three consecutive integers a multiple of 6? How does factoring a quadratic connect to factoring a number?”

Connecting to areas outside of mathematics requires teachers to have a disposition to look for mathematics outside of their classroom, both within and beyond the boundaries of the school walls and to seek mathematical explanations for real world quantitative relationships. Today’s electronic technology offers multiple opportunities for such explorations. For example, video games employ matrix operations to animate images on the screen through geometric transformations. Designers of automobiles use Bezier curves to render pictures of new designs for cars. Teachers who introduce ideas such as these to students can not only engage them in simulating what those who use mathematics in real life do but also help them to understand the mathematical ideas shaping the world in which they live. Statistical considerations are another robust area for quantitative exploration. For example, it is interesting to know that The Federalist

Papers were written by three different people using the pseudonym Publius, and statistical tests have been used to estimate who authored which paper. The point is not that there is some ordained list of applications that a teachers needs to know, but rather that there are intriguing topics that teachers can explore with their students by applying mathematics at a secondary level and that teachers should be willing and able to seek out the resources to investigate these topics. Connecting within and outside mathematics means looking for applications of mathematics as well as situations from which to extract mathematics. Every teacher may not need to know something about a particular connection, but all teachers need to know the properties of the mathematical entities about which they are teaching well enough to recognize an application when they see it. This recognition involves seeing the properties of the mathematical entities well enough to match them to the situation (Zbiek & Conner, 2006). Identifying and matching these properties are the action the teacher must own.

Represent. Inherent in the task of teaching is the need to create representations for mathematical entities from given structures, constraints, or properties. At all levels of mathematics, teachers represent mathematical objects using different types of representations. For each representation they use, teachers communicate about it and interpret it in the context of what it signifies, orchestrate movements between representations, and craft analogies to describe representations, objects, and relationships. They describe mathematical objects using numbers, symbols, pictures, words, physical objects, and other means. It is important for teachers not only to be adept at creating and interpreting representations but also to recognize that students may view the representation as the mathematical object of interest: Students may view the graph of a linear function as the line, and they may view the sketch of a circle as the circle itself. For example, when one teacher asked his students whether it was possible for a linear function to have neither x -intercepts nor y -intercepts, one of his students replied that it was possible. The student had drawn the axes with short line segments and had observed that the line segment representing the line did not intersect either segment representing the axes. The student was treating the representation of the lines involved as if they were the lines themselves.

Teachers also need to be able to analyze each new representation, recognizing what features it captures of the mathematical object and what features it does not capture. They need to be able to help students develop a critical eye in choosing and interpreting representations and in recognizing how a feature in one representation is related to what may appear to be a completely different feature in another representation. Each representation affords different views of the mathematical object, but several different representations can highlight the same feature. Teachers need to develop a repertoire of representations that are helpful in answering particular kinds of questions: For example, a verbal representation such as SOHCAHTOA may help students remember the ratios for different trig ratios, and a short description such as “slope is rise over run” may help students calculate the slope for a given linear graph. Teachers also need to recognize that a given feature can be interpreted in different ways; for example, seeing the “vertical line test” to determine whether a relation is a function as an indicator of the unique output requirement of a function instead of simply a trick to help answer questions about functions (e.g., students viewing the graph of $x = y^2$ may

claim that it is not a function without entertaining the possibility that x could be the dependent variable).

Teachers who can represent well should be able to switch smoothly between representations and know that each representation emphasizes different aspects of the same object. With access to a broad range of representational forms, teachers can use or create equivalent representations to reveal different information or to foreground a particular concept (e.g., looking at table rather than graph to quantify rate of change; looking at derivative for rate of change in a numerical situation; looking at the graph of a function to highlight local extrema). Teachers' repertoires of representations help them select models for given number types or operations (e.g., multiplication of rational numbers, addition of complex numbers). In a geometry class, a teacher might use physical objects for quadrilaterals and other shapes while also having students explore what conditions must be met to create similar shapes on a program such as Geometer's Sketchpad (GSP). Teachers use analogies and language to describe functions as well, using function machines or other analogies to impart some of the qualities of a function. Teachers represent numbers in different mathematical settings. For example, the mathematical meaning of $\frac{a}{b}$ (for real numbers a and b , with $b \neq 0$) arises in several different mathematical settings, including slope of a line, direct proportion, Cartesian product, factor pairs, and area of rectangles. In explaining a definition of $\frac{a}{b}$, a teacher might choose slope of a line as a setting to illustrate the need for $b \neq 0$. In each representational setting, teachers need to be able to use mathematically precise language to communicate to students about the representations the students generate as well as about the ones they are given.

A popular description of the modeling process starts with a real-world problem that is translated into a formal mathematical system. Within the formal system, the model is manipulated until a solution is found. The solution is mapped back to the real world to be tested with the problem. Schoenfeld (1994) points out that the validity of the ensuing analysis depends on the accuracy of both of the mappings to and from the formal system. It is important to note that the issue is one of fit rather than absolute correctness. Modeling can be seen as a recursive process. If a solution does not fit the real-world context well enough, aspects of the model, such as initial conditions that are assumed, could be changed to form a new model. Programs such as GSP allow geometrical models to be created to test hypotheses. Statistical modeling provides predictions when dealing with data points. Monte Carlo simulations model outcomes using random inputs. Note that we see modeling as involving a context outside of mathematics as distinct from representing, which resides wholly within mathematics.

Secondary mathematics teachers who know the difference between modeling a situation and applying a piece of mathematics to a realistic situation and who also know that what constitutes a "good" or "good enough" model depends on the setting will be better able to engage their students in the modeling process than teachers lacking such knowledge. Every secondary mathematics teacher would benefit from understanding modeling as involving both mathematical and statistical concepts. Zbiek and Conner

(2006) describe the action of mathematical modeling as including the processes of *specifying* (“identifying the *conditions* and *assumptions* ... of the real-world context to which the modeler will attend as he or she mathematizes the situation” [p. 99]), *mathematizing* (creating or acknowledging mathematical *properties* and *parameters* ... that correspond to the situational conditions and assumptions that have been specified” [p. 99]), *interpreting* (“putting the mathematical conclusion in context” [p. 103]), and *examining* (“comparing the real-world conclusion with the situation while considering the modeling purpose to ensure the real-world conclusion aligns with the realistic situation in light of the modeling goal” [p. 104]). Teaching secondary students about mathematics in real-world settings requires that teachers recognize the properties of mathematical entities in such situations. They should be able to see a property of a mathematical entity in a situation, they should notice characteristics of situations that they can associate with a mathematical entity, and they should be able to judge the reasonableness of a mathematical result within a context.

Constrain and extend. Cuoco (1996) argues that “mathematicians talk small and think big.” Teachers who generalize are able to test conjectures, expand the domains of rules and procedures, and adapt mathematical ideas to new situations. If a conjecture is made in a classroom, a teacher will be able to test the conjecture with different domains or sets of objects. For example, a student may state that multiplication returns a number equal to or larger than either initial factor. A teacher should be able to test whether such a conjecture holds true in every domain. Similarly, a teacher should be able to explain why rules may or may not work in new domains. Although a teacher might demonstrate the exponent rules with rational exponents, the teacher should also be able to demonstrate why such rules still work for complex exponents.

Teachers are frequently called on to constrain or extend the domain, argument, or class of objects for which a mathematical statement is or remains valid while preserving the structure of the mathematical statement (e.g., extending the concept of absolute value to a modulus definition as the domain is extended from real to complex numbers; extending the object “triangle” from Euclidean to spherical geometry). They should be able to interpret certain mathematical conditions or constraints that are relevant to a mathematical activity and recognize when it is useful to relax or constrain mathematical conditions (e.g., recognizing that it is not true that any number raised to the n th power is equivalent to 1). They make mathematical generalizations by extending the domain to which a set of properties apply, thus identifying a larger set of instances to which the properties apply. With these capacities, teachers can create mathematical extensions to given problems and questions and can recognize the implications of a extending or constraining the domain, argument, or class or objects under consideration.

With secondary mathematics as the bridge between prealgebra mathematics and collegiate mathematics, secondary mathematics teachers are often challenged to explore the consequences of imposing or relaxing constraints. To *constrain* in mathematics means to define the limits of a particular mathematical idea. When finding the inverses of a function, one must sometimes constrain the domain if one wants the inverse to be a function as well. The inverse of $f(x) = \sin x$ is a function only if the new domain is

restricted. Constraints can be removed or replaced to explore the resulting new mathematics. When mathematicians tinkered with the constraint of Euclid's fifth postulate, new geometries were formed. When one removes the constraint of the plane in using Euclidean figures, the mathematics being used changes as well. Secondary mathematics teachers regularly encounter situations in which to provide a suitable response, they must tailor a generalization so that it can reasonably be extended to a larger domain of applicability. For example, they may have to analyze the extent to which properties of exponents generalize from natural number exponents to rational number or real number exponents, or they may find it useful to generate an example of a situation for which multiplication is not commutative. Teachers with an understanding of the mathematics their students will encounter in further coursework can structure arguments so that they extend to a more general case. For example, teachers who recognize that a graphical approach to solving polynomial equations is far more generalizable than the usual set of polynomial factoring techniques may tend to provide their students with a more useful technique.

Another example of constraining occurs when teachers constrain the domain within which the class is to work. Some geometric proofs are simple in coordinate geometry. However, if the teacher constrains all proving to synthetic geometry, then techniques must be used that display different mathematics than would be seen if the proof were performed only one way.

Teachers' daily work involves them in creating and using examples, nonexamples, and counterexamples for mathematical objects, generalizations, or relationships. The creation of counterexamples requires knowing the properties of the concepts involved in a proposed generalization and variations of those properties so that a concept can be selected for which the generalization is not true. For example, the generalization that multiplication is always commutative can be shown to be false when multiplication is defined over matrices—this is also an example of the process of constraining or extending. Creating examples and nonexamples requires understanding the properties required and the implications of those properties. For example, creating a polynomial that does or does not factor over the reals is assisted by knowing solutions to polynomials of degree four or higher. Similarly, the ability to create a graph with particular characteristics is helped by knowledge of what the derivative and extrema of a symbolically stated function indicate about the shape of its graph. Teachers' work is enabled by their ability to choose examples that serve their purposes as well as their ability to generate specific examples from a set of conditions or from an abstract idea.

Secondary mathematics teachers are constantly called on to apply and generate mathematical generalizations, and they need to evaluate the truth of generalizations that their students propose. The process of generalizing is intimately related to constraining and extending. Generalizing is the act of extending the domain to which a set of properties apply from multiple instances of a class or from a subclass to a larger class of mathematical entities, thus identifying a larger set of instances to which the set of properties applies. Students often state generalizations in an overly broad way, and so their teachers need to be adept at constraining the domain of proposed generalizations so as to make the statement true. Secondary mathematics teachers need to be able to

articulate needed constraints in ways that build on their students' understanding. It will not help students to hear the most accurate revision of their generalization if that revision is stated in terms of mathematical entities with which they are not familiar.

Justify and prove. Teaching mathematics well requires justifying mathematical claims through logical connections or deductions among mathematical ideas. Teachers of secondary mathematics need to be comfortable with a range of strategies for mathematical justification, including both formal justification and informal arguments. Formal justification, or proof, requires basing arguments on a logical sequence of definitions, axioms, and known theorems. Teachers need a different sort of justification ability than other users of mathematics. They need to be able to understand and formulate different levels and types of mathematically and pedagogically viable justifications and proofs. It is not always the case that claims to be proved arrive on a teacher's desk fully formed—the evidence and claim may originate from a student's observations. In this and other settings, secondary mathematics teachers are called on to consider or generate empirical evidence, formulate conjectures from that evidence, and prove or disprove those conjectures deductively. They need to be able to formulate and structure their arguments at a range of appropriate levels, they need to be able to state assumptions on which a valid mathematical argument depends, and they need to recognize the need to specify assumptions in an argument.

Positioning themselves to accomplish all of the above requires of secondary mathematics teachers a breadth of experience and expertise with justification as well as with proof. Teachers need to be on the alert for special cases (e.g., any number raised to the zero power is not always 1), they need to recognize an exhaustive list of cases, and they need to recognize the limitations of reasoning from diagrams. Not only do they need experience with justification and proof, but they also need to be able to craft explanations that communicate aspects of justification at an appropriate level. For example, they need to be able to explain why a process does not generalize when applied to a different entity, and they need to be able to explain the logic or organizing idea of a formal proof to students without extensive (or any) experience in constructing proofs. Examples arising from our situations include:

1. Constructing an array of explanations for why the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, including appealing to cases, by making strategic choices for pair-wise grouping of numbers and by appealing to arithmetic sequences and properties of such sequences.
2. Arguing by contradiction (excluded middle): To prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle, construct a circumcircle about three vertices of a quadrilateral and argue that if the fourth vertex can be in neither the interior nor the exterior of the circle, then the fourth vertex must be on the circumcircle, and therefore the quadrilateral can be inscribed in a circle.

Mathematical Work of Teaching

The mathematical work of teaching requires that teachers be able to help someone else know and do mathematics. In Ryle's (1949) terminology, the mathematical work of teaching requires both knowing how and knowing that. It moves beyond the goal of establishing a substantial and continually growing proficiency in mathematics for oneself as a teacher to include the goal of effectively helping one's students develop mathematical proficiency.

Not only should teachers of secondary mathematics be able to know and do mathematics themselves, but also their proficiency in mathematics must prepare them to facilitate their students' development of mathematical proficiency. Possessing proficiency in the mathematical work of teaching mathematics enables teachers to integrate their knowledge of content and knowledge of processes to increase their students' mathematical understanding.

Probe mathematical ideas. The first category of the mathematical work of teaching addresses the type of knowledge that is useful for investigating and pulling apart mathematical ideas. Mathematics is dense. One goal in doing mathematics is to compress numerous complex ideas into a few succinct, elegant expressions. These expressions can be used to build additional ideas that will also become compressed. Although mathematical efficiency and rigor are essential if one is to engage in complex mathematical thinking, they can also cause confusion, especially for those just being initiated into the culture of mathematics.

Teachers need to be able to see complexity in simple ideas and also be able to reduce the complexity of mathematical ideas without destroying their integrity—a challenging mathematical feat. That is, they need to be able to reverse the process of compression of ideas. Consider how a teacher might begin the study of complex numbers with students who are comfortable working with real numbers. They may think, “You cannot take the square root of a negative number.” It may not enter their thinking that it is possible to have a number system that is not well ordered, and they may believe that all numbers fit somewhere on a one-dimensional number line. The students may have been told this information by their teachers or colleagues, or they may have inferred it from their previous mathematical work. It is difficult, and perhaps not wise, for a teacher to provide explicit information about every aspect of a concept. It would be absurd for young students trying to comprehend the notion of multiplication of whole numbers, for example, to be warned that multiplication of matrices does not exhibit the same properties. But teachers need to have mathematical knowledge that includes a deep knowledge of multiplication and multiplicative structures so they can identify essential aspects of multiplication that are appropriate for their students, and that do not interfere with the expansion of the concept.

Probing mathematical ideas requires a broad knowledge of mathematical content and associated mathematical activities such as defining, representing, justifying, and connecting. Teachers need mathematical knowledge that will help them to pull apart mathematical ideas in ways that allow the ideas to be reassembled as students mature

mathematically. They need to recognize and honor the conventions and structures of mathematics and recognize the complexity of elegant mathematical ideas that have been compressed into simple forms.

Access and understand the mathematical thinking of students. The second category refers to knowledge that helps teachers understand how their students are thinking about mathematics. Accessing students' thinking is quite different from the evaluative process of assessing students' understanding. A proficient teacher uncovers students' mathematical ideas, seeing the mathematics from a learner's perspective. Teachers can gain some access to students' thinking through written work they do in class or at home, but much of that information is highly inferential. Through discourse with students about their mathematical ideas, the teacher can learn more about the thinking behind their written products. Communication among students and between them and their teacher is vital for developing their mathematical thinking and for helping their teacher shape that thinking. Classroom interactions play a significant role in teachers' understanding of what their students know and are learning. It is through a particular kind and quality of discourse that implicit mathematical ideas are exposed and made more explicit.

Building a practical understanding of and knowledge base of actions for engaging students in discourse about important mathematical ideas requires a specialized knowledge of mathematics. Students often discuss mathematics using vague explanations or terms that have a colloquial meaning different from their mathematical meaning. A teacher needs the proficiency to probe informal explanations, help students focus on essential mathematical points, and help them learn conventional terms. Success in such endeavors requires understanding the nuances and implications of students' understanding and recognizing what is right about their thinking as well as features of their thinking that lead them to unproductive conceptions.

At the same time, the teacher needs to avoid discouraging or distorting the students' mathematical thinking, which often begins with vague, imprecise explorations. Achieving such a balance requires the teacher to have an extensive knowledge of mathematical terminology, formal reasoning processes, and conventions, as well as an understanding of differences between colloquial uses and mathematical uses of terms. For example, in a class discussion of Platonic solids, a student might propose a conjecture about the number of sides and number of vertices. Some students may interpret *sides* to mean *faces*; others may be thinking *edges*. A teacher who knows and anticipates such potential confusion can use it to motivate the class to reject the imprecise term *side* and define the terms *face* and *edge*. This elaboration can be handled without losing sight of the valuable conjecture made by the student.

Know and use the curriculum. The third category refers to the mathematical knowledge that helps teachers know and use the curriculum to help students connect mathematical ideas and progress to a deeper and better grounded mathematics. How mathematical knowledge is used to teach mathematics in a specific classroom or with a specific learner or specific group of learners is influenced by the curriculum that organizes the teaching and learning. A teacher's mathematical proficiency can help

make that curriculum meaningful, connected, relevant, and useful. For example, a teacher who is proficient in the mathematical work of teaching may have a perspective on the curriculum for the concept of area that includes ideas about measure, descriptions of two-dimensional space, measures of space under a curve, measures of the surface of three-dimensional solids, infinite sums of discrete regions, operations on space and measures of space, foundations of the geometric properties of area, and useful applications involving area. This perspective on the curriculum is very different from that of someone who thinks of area only in terms of formulas for polygonal regions.

Proficiency in knowing and using the mathematics curriculum in teaching requires a teacher to identify foundational or prerequisite concepts that enhance the learning of a concept as well as how the concept being taught can serve as a foundational concept for future learning. The teacher needs to know how the concept fits each student's learning trajectory. The teacher also needs to be aware of common mathematical misconceptions and how those misconceptions may sometimes arise from instruction. Proficient mathematics teachers understand that there is not a fixed order for learning mathematics but rather multiple ways to approach a mathematical concept and to revisit it. Mathematical concepts and processes evolve in the learner's mind, becoming more complex and sophisticated with each iteration. Mathematical proficiency prepares a teacher to enact a curriculum that not only connects mathematical ideas explicitly but also develops a disposition in students so that they expect mathematical ideas to be connected and an intuition so that they see where those connections might be (Cuoco, 2001).

A teacher proficient in the mathematical work of teaching understands that a curriculum contains not only mathematical entities but also mathematical processes for relating, connecting, and operating on those entities (National Council of Teachers of Mathematics, 1989, 2000). A teacher must have such proficiency to set appropriate curricular goals for his or her students (Adler & Davis, 2006). For example, a teacher needs special mathematical knowledge to select and teach functions in a way that helps students build a basic repertoire of functions (Even, 1990).

Assess the mathematical knowledge of learners. The fourth category concerns knowledge that enables the teacher to assess or evaluate students' mathematical understanding. Teachers need to recognize student errors and be able to analyze those errors to see how they are related. They need to be able to design tasks that will assist them in evaluating the students' depth of understanding. All students bring their own previous experiences into a mathematics class, and each lesson provides them with opportunities to develop their mathematical knowledge through a variety of learning tasks. Each task also provides the teacher with opportunities to observe what students understand and to identify their errors. Assessing students' mathematical knowledge involves much more than assessing a student's ability to follow a procedure. The teachers' knowledge of the mathematical work of teaching should help them identify essential components of mathematical concepts, enabling them to create tasks that assess students' understanding and ability to use and connect mathematical ideas.

Many errors arise from students' failure to appreciate the consequences of expanding or constraining the set of elements under consideration. Teachers may fail to make the domain of a function explicit or to point out different usages of a term. For example, students often confuse finding the inverse of a function with finding its reciprocal, or multiplicative inverse. Assessing and analyzing such confusion may not only help students correct a common error but also highlight the problem of not explicitly identifying the operation (composition) associated with finding the inverse function. Also, students' limited understanding of a procedure may inhibit increasing that understanding when the set of elements over which the procedure has been defined and used is extended. Treating exponentiation as repeated multiplication, for example, may create conflict for students when they are asked to consider a number raised to a fractional or irrational power.

Proficiency in assessing the mathematical knowledge of learners requires that teachers understand the structure of mathematical systems and are aware of what needs to be made explicit in class discussions and tasks. Without such understanding and awareness, a teacher may assume that because students happen to be using mathematical terms correctly, they know the structures and operations to which the terms apply.

Reflect on the mathematical problems of practice. The fifth category concerns knowledge that enables teachers to analyze their mathematics teaching in a way that leads to enhancing their own mathematical knowledge. There are many ways to reflect upon one's teaching, but it is important to reflect on the work of teaching through a mathematical lens. Did the tasks help students focus on the core mathematical ideas? Did I use conventional mathematical vocabulary and notation? Why or why not? What did my instruction imply about the nature of mathematics? How can I connect my lesson to previously learned mathematics? Thoughtful reflection on problems of practice can be reconsideration of a lesson just taught, or it can be part of the planning for a future lesson. It may occur as the teacher interprets the results of a formal assessment, or it may be prompted by a textbook treatment of a topic. Whatever its origin, reflection needs to include thinking about the mathematics of the lesson as well as its pedagogical features.

Teachers are often reflecting about their teaching as they teach—as they are making split-second mathematical and pedagogical decisions. A teacher's decision about how to proceed after accessing student thinking depends on many factors, including the mathematical goals of the lesson. It is valuable to revisit these quick reflections and decisions when there is time to think about how particular problems of practice might inform future teaching. The goal is not to avoid mathematical problems while teaching but rather to reflect on how such problems can lead to better teaching and better mathematical understanding.

The framework for MPT has been developed over the past few years. We began with the idea of designing a framework for the construct of *mathematical knowledge for teaching* (MKT) as it might be applied to secondary school mathematics. Working from the bottom up, we began by developing a collection of sample situations. Each situation portrays an incident in teaching secondary mathematics in which some mathematical point is at issue. (For details of our approach, see Kilpatrick, Blume, & Allen, 2006.) Looking across situations, we attempted to characterize the knowledge of mathematics that is beneficial for secondary school teachers to have but that other users of mathematics may not necessarily need.

Our initial characterization was much influenced by the work of Deborah Ball and her colleagues at the University of Michigan (Ball, 2003; Ball & Bass, 2000; Ball, Bass, & Hill, 2004; Ball, Bass, Sleep, & Thames, 2005; Ball & Sleep, 2007; Ball, Thames, & Phelps, 2008). In particular, Ball et al. have partitioned MKT into components that distinguish between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). They have identified four components: common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Ball et al., 2004). And more recently, they have added two additional kinds of knowledge: knowledge of curriculum and knowledge at the mathematical horizon. An example of the latter is “being aware that two-digit multiplication anticipates the more general case of binomial multiplication later in a student’s mathematical career” (Ball, 2003, p. 4). Figure 3 shows the six components and how they are related.

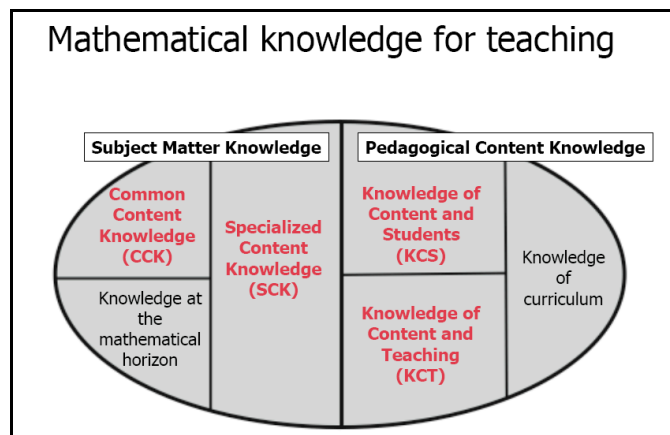


Figure 3. Model of MKT (Ball & Sleep, 2007).

As we worked on developing our own framework, we considered attempts to develop similar frameworks (e.g., Adler & Davis, 2006; Cuoco, 1996, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tatto et al., 2008). We became increasingly concerned that whatever framework we developed needed to reflect a broader, more dynamic view of mathematical knowledge.

The current version of the framework is [Need to finish this point to acknowledge the various sources of input. But first we should check that the above is what we want to say.]

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